- 分离变量 $y'=xy \rightarrow \frac{dy}{dx}=x\cdot y$

  (x,y乘除,能分开) $\frac{1}{y}dy = x\,dx$

  $$\ln|y| = \frac{1}{2}x^2+C$$

  $$y = \pm Ce^{\frac{1}{2}x^2}$$

- 一阶线性方程 $y'+py=q$

  通解 $y_n = Ce^{-\int p\,dt}$

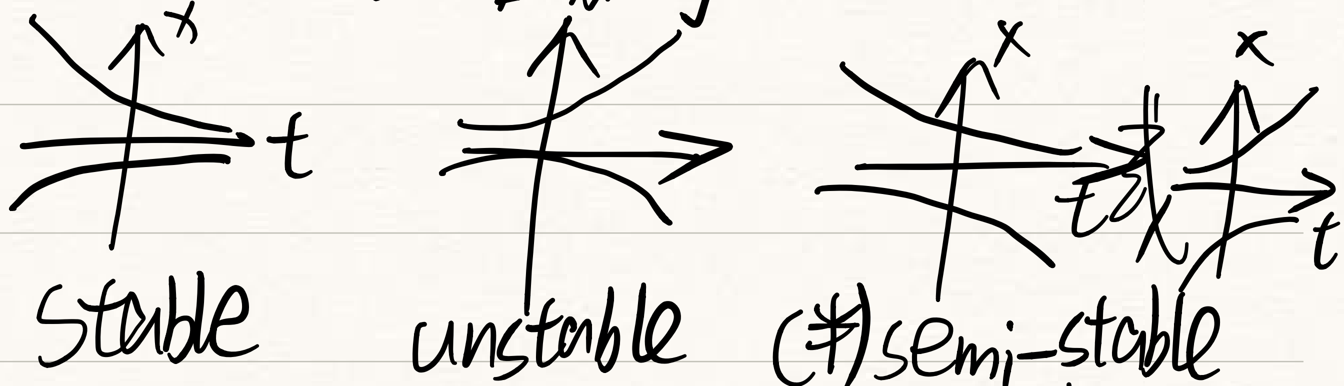
  y_p 那令 $C=$ 函数 v
  代回原方程得

  特解 $y_p = e^{-\int p\,dt}\cdot\int q(t)\cdot e^{\int p\,dt}\,dt$

  $$y = y_n + y_p$$

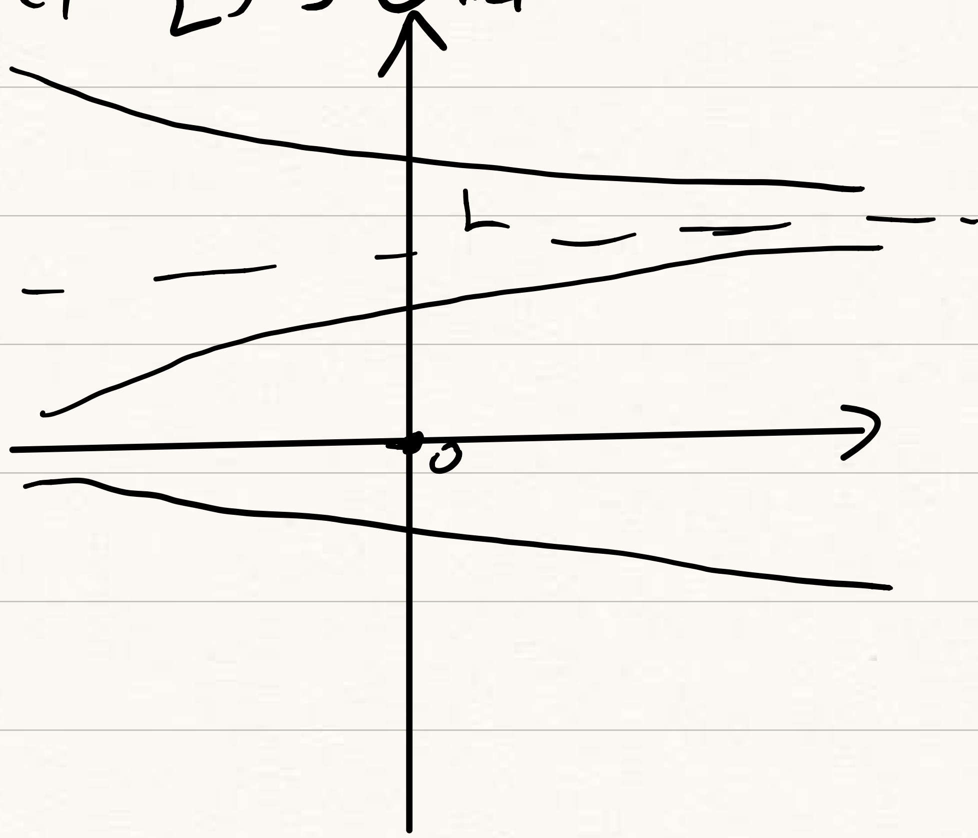
- 不含t的(仅绘制图像)

  $x'=f(x)$，已知$f(x)$图像[illegible] $x(t)$走向 不考

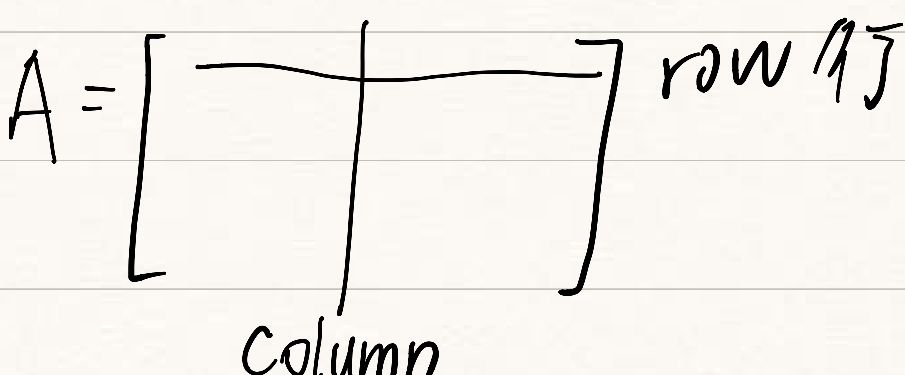


当 $y=0$, $y=L$ 时 $y'=0$

已知 $y' = k(1-\frac{y}{L})\cdot y$ 画图



• 矩阵



列

@转置 T

$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

求导

$$\frac{d\begin{bmatrix} e^t, & \ln t, & \cos t \\ t^2, & t\sin t, & 2^t \end{bmatrix}}{dt} = \begin{bmatrix} e^t, & \frac{1}{t}, & -\sin t \\ 2t, & \sin t + t\cos t, & 2^t \ln 2 \end{bmatrix}$$

@基本乘法

@公式 $AB \neq BA$

$$(AB)C = A(BC)$$

$$(A+B)C = AC + BC$$

@单位阵 $I_n = \begin{bmatrix} 1 & & \\ & 1 & \\ & & \ddots \end{bmatrix}$

$I_n \cdot A$ 或 $A \cdot I_n$ 都等于 A

Pivots 主元

pivots 主元 $\begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 9 \end{bmatrix}$

form
- echelon 行阶梯型 $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ 若有零行则在下
- reduce row echelon form (RREF) 行最简型 打头为 1 主元
- not/non- echelon 非行阶梯型 $\begin{bmatrix} 5 & 1 & 1 \\ 1 & -4 & 7 \\ 0 & 0 & 1 \end{bmatrix}$

@ 增广矩阵 Augmented Matrix

$$[A|\vec{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

$A\vec{x} = \vec{b}$

1) 可交换两行

2) 可作0以外的行倍乘

3) 两行可作相加 $R_i^* = R_i + cR_j$

Ex: solve

$$\begin{cases} x+z=2 \\ 2x+3y+5z=4 \\ 3x+2y-z=2 \end{cases} \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 2 & 3 & 5 & 4 \\ 3 & 2 & -1 & 2 \end{array}\right]$$

$$\xrightarrow[R_3^*=R_3-3R_1]{R_2^*=R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 3 & 3 & 0 \\ 0 & 2 & -4 & -4 \end{array}\right] \xrightarrow[R_3^*=\frac{1}{2}R_3]{R_2^*=\frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -2 & -2 \end{array}\right] \xrightarrow[R_3^*=R_2-R_3]{R_2^*=2R_2+R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 3 & 0 & -2 \\ 0 & 0 & 3 & 2 \end{array}\right]$$

$$\xrightarrow[R_3^*=\frac{1}{3}R_3]{R_2^*=\frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & \frac{2}{3} \end{array}\right] \xrightarrow{R_1^*=R_1-R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{4}{3} \\ 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & \frac{2}{3} \end{array}\right]$$

1) 零行置底

2) 每非零行最左边的非零数叫主元 pivot

3) 每一个主元都比上面一行更右 (阶梯乙)

4) 含主元的列叫主元列 pivot column.

行最简型

Ex: is each matrix before RREF?

$$\begin{bmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{bmatrix} \qquad \begin{bmatrix} 1&0&1\\ 0&1&-2\\ 0&0&0 \end{bmatrix} \qquad \begin{bmatrix} 1&1\\ 0&1\\ 0&0 \end{bmatrix} \rightarrow \begin{bmatrix} 1&0\\ 0&1\\ 0&0 \end{bmatrix}$$

4 pivots
4 pivots column
RREF

2 pivots
2 pivots column
RREF

non-RREF RREF

Gauss Jordan elimination 高斯消元法

☆ $$\left[\begin{array}{ccc|c} 1&1&1&1\\ 1&2&1&4\\ 2&3&2&3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1&1&1&1\\ 0&1&0&3\\ 0&1&0&0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1&0&1&-2(0)\\ 0&1&0&3\\ 0&0&0&-3 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1&0&1&-2(0)\\ 0&1&0&3\\ 0&0&0&1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1&0&1&0\\ 0&1&0&0\\ 0&0&0&0 \end{array}\right]$$

$$\begin{cases} A^{-1} & \text{逆矩阵（方阵，行最简式，主元=n）} \\ A^{T} & \text{转置} \end{cases}$$

A^{-1}逆矩阵定义为 对A有B使$AB=BA$
则B为A的逆矩阵，设A^{-1}为B

找A^{-1} inverse matrix

1) 硬算

Ex Find A^{-1} for $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$

设 $A^{-1}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ 则 $A\cdot A^{-1}=\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\cdot\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

得 $\begin{bmatrix} a+c & b+d \\ 2a+c & 2b+d \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{cases} a+c=1 \\ 2a+c=0 \end{cases}$ 得 $a=-1$，$c=2$

$\begin{cases} b+d=0 \\ 2b+d=1 \end{cases}$ 得 $b=1$，$d=-1$

$$A^{-1}=\begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$$

2) 增广矩阵|单位阵变位 令 $[A|I_n] \rightarrow [I_n|A^{-1}]$

Ex2: find the C^{-1} of $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3^*=R_1-R_3]{R_1^*=R_1,\ R_2^*=R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \end{array}\right]$$

$$\xrightarrow[R_3^*=R_2]{R_2^*=R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 2 & 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_3^*=R_3-2R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{R_1^*=R_1-R_2-R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{array}\right] \text{ get } C^{-1} = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ -2 & 1 & 2 \end{bmatrix}$$

验证: $CC^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 证毕

@ 行列式 determinant 求解

记 det(A) 矩阵A的行列式

1) 2X2 $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

$= a_{11}a_{22} - a_{12}a_{21}$

2) 3X3 $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$ $(-1)^{m+n}$ 系数为负数 a_{13} 有 $1+3=4$ 为正数

$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

$= -a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$

(inverse 反转，逆阵，有时会让判断)

ⓐ 行列式性质

1: $\det I_n = |I_n| = 1$

2: 矩阵交换行，行列式取反(负数)

3: 矩阵一行乘k，行列式(结果)乘k

$\Downarrow$

④ 两行一样，(差倍数) 行列式为零
(交换后行列式相同且相反，为0)

⑤ 高斯消元，行列式不变 对自己不能倍数减其它行，其它行可倍数

⑥ 矩阵有零行，行列式为零

⑦ 三角阵行列式为对角线上乘积

$$\begin{vmatrix} d_1 & \cdots & \cdots \\ & d_2 & \cdots \\ & & d_3 \cdots \end{vmatrix} \xrightarrow{\text{消元}} \begin{vmatrix} d_1 & 0 & 0 \cdots \\ & d_2 & 0 \cdots \\ & & d_3 \; 0 \cdots \end{vmatrix} = d_1 d_2 d_3 \cdots$$

⑧ 若 A 有逆阵 A^{-1}，则 $|A^{-1}| = \dfrac{1}{|A|}$
invertible

当且仅当($\Longleftrightarrow$) $|A| \neq 0$ ($\Longleftrightarrow$ 满秩)

⑨ $|A^T|=|A|$ ($\begin{vmatrix}a&b\\c&d\end{vmatrix}=ad-bc=\begin{vmatrix}a&c\\b&d\end{vmatrix}$)

列交换与行交换相同，行列式转置值不变

inverse

⑩ $\det(AB)=\det A\cdot\det B$

有 $\det A^{-1}=\frac{1}{\det A}$ 则 $\det I=\det A\cdot\det A^{-1}=1$

⑪ 矩阵加法全叠加，行列式加法某行叠加

$$\begin{vmatrix}a+a' & b+b'\\ c & d\end{vmatrix}=\begin{vmatrix}a&b\\c&d\end{vmatrix}+\begin{vmatrix}a'&b'\\c&d\end{vmatrix}$$

ⓐ $A\vec{x}=\vec{b}$ 求解 x

1) 若 A^{-1} 存在

$\vec{x}=A^{-1}\vec{b}$ 存唯一解

克莱母 Cramel 法则

$$\vec{x}=\begin{vmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{vmatrix}$$

$$\vec{x}_1=\frac{|A_1|}{|A|} \quad \vec{x_2}=\frac{|A_2|}{|A|} \quad \vec{x_3}=\frac{|A_3|}{|A|}$$

Ex $\begin{cases} x_1+x_2+x_3=1 \\ x_1-x_2+5x_3=6 \\ -x_1+x_2+6x_3=9 \end{cases}$ $A=\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 5 \\ -1 & 1 & 6 \end{vmatrix}$

$A_1=\begin{vmatrix} 1 & 1 & 1 \\ 6 & -1 & 5 \\ 9 & 1 & 6 \end{vmatrix}$ $A_2=\begin{vmatrix} 1 & 1 & 1 \\ 1 & 6 & 5 \\ -1 & 9 & 6 \end{vmatrix}$

$A_3=\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 6 \\ -1 & 1 & 9 \end{vmatrix}$

@向量空间和子空间

Vector space and subspace

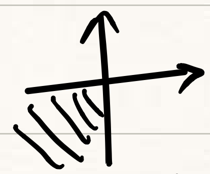
一列 column A : $a \in A$

如何判断V的子空间H

1、加法封闭 closed under addition

2、数乘封闭 closed under multiplication

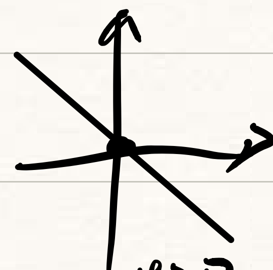
Ex



加法封闭

数乘不封闭，乘"-1"就在第1象限了

Ex



加法封闭

数乘封闭

Ex: 已知 $\vec{u}=\begin{bmatrix}1\\1\\0\end{bmatrix}$ $\vec{v}=\begin{bmatrix}-1\\0\\2\end{bmatrix}$

find span$\{\vec{u},\vec{v}\}$

$$c\begin{bmatrix}1\\1\\0\end{bmatrix}+d\begin{bmatrix}-1\\0\\2\end{bmatrix}=\begin{bmatrix}c-d\\c\\2d\end{bmatrix} \text{ for all } c,d\in\mathbb{R}$$

$$x=c-d \quad y=c \quad z=2d$$

$$\boxed{x=y-\frac{z}{2}}$$

span$\{\vec{u},\vec{v}\}$ is plane $x-y+\frac{z}{2}=0$

@ 线性独立 linear Independence

满秩，

若有重行则某变量无限解

$v_1\, v_2 \ldots v_k$ are independent

so, $v_1\, v_2 \ldots v_k$ span the Vector space

可消元看是否独立(满秩)

Ex: 已知 $\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $\vec{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\vec{u}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$\vec{b} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$

1. is $\vec{b}$ in the span $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$?

$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \vec{b}$ 看是否存在

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}=\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$$

可求逆阵也可求增广阵解

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 2 & 4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad \begin{matrix} c_1=1 \\ c_2=1 \\ c_3=2 \end{matrix}$$

$\vec{b}=\vec{u_1}+\vec{u_2}+2\vec{u_3}$ 则 $\vec{b}\in \text{Space}\{\vec{u_1},\vec{u_2},\vec{u_3}\}$

空间性质Fact:

1) 任何 $\mathbb{R}^n$ (any spanning set of $\mathbb{R}^n$) 至少有 n 个向量，并且 n 个独立

$\Leftrightarrow$ 行列式 $[\vec{u_1}|\vec{u_2}|\dots|\vec{u_n}]\neq 0 \Leftrightarrow$ 满秩

2). 线性独立、非线性独立

linear independence \ dependence

给定向量组 $\vec{v_1}$、$\vec{v_2}$, …, $\vec{v_n}$ 在 V 空间中线性相关(不独立)

则有一组 c_1、c_2, …, c_n (不全为零) 能使 $c_1\vec{v_1}+c_2\vec{v_2}+\dots+c_n\vec{v_n}=0$

3) 若系统有无限多个解 (infinity many) 则 线性相关 (有零行,不满秩)

4) 若系统仅有零解一个，则为线性独立 (∵满秩)

EX: Are the vector $\begin{bmatrix}1\\1\\1\end{bmatrix}\begin{bmatrix}-1\\0\\1\end{bmatrix}\begin{bmatrix}-2\\1\\4\end{bmatrix}$ linearly dependent or linearly independent? If dependent, find the relationship.

先 $\begin{vmatrix}1 & -1 & -2\\1 & 0 & 1\\1 & 1 & 4\end{vmatrix} \ldots = 0$ 知非线性独立

$$c_1\begin{bmatrix}1\\1\\1\end{bmatrix}+c_2\begin{bmatrix}-1\\0\\1\end{bmatrix}+c_3\begin{bmatrix}-2\\1\\4\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}1 & -1 & -2 & 0\\1 & 0 & 1 & 0\\1 & 1 & 4 & 0\end{array}\right] \rightarrow \left[\begin{array}{ccc|c}1 & -1 & -2 & 0\\0 & 1 & 3 & 0\\0 & 2 & 6 & 0\end{array}\right] \rightarrow \left[\begin{array}{ccc|c}1 & 0 & 1 & 0\\0 & 1 & 3 & 0\\0 & 0 & 0 & 0\end{array}\right]$$

令 $c_3=1$ 解 $c_2=-3$, $c_1=-1$

即 $-\begin{bmatrix}1\\1\\1\end{bmatrix}-3\begin{bmatrix}-1\\0\\1\end{bmatrix}+\begin{bmatrix}-2\\1\\4\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

5、线性独立的含参方程 e^t（与前面一致）

Linear independence of function

$\vec{v}_1(t), \ldots, \vec{v}_k(t)$, $t \in I$

with value in R^m are dependence if there exist $c_1, \ldots, c_k$ scalars, (not all zero), so that $c_1\vec{v}_1(t) + \ldots + c_k\vec{v}_k(t) = \vec{0}$, for all $t \in I$

Ex: $\vec{v}_1(t) = \begin{bmatrix} e^t \\ e^t \end{bmatrix}$ $\vec{v}_2(t) = \begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix}$, $t \in R$

values in R^2

$$\begin{vmatrix} e^t & e^{2t} \\ e^t & -e^{2t} \end{vmatrix}$$

$= -e^{3t} - e^{3t} = -2e^{3t} \neq 0$

则线性独立

at least one t value that $\vec{v}_1(t), \vec{v}_2(t)$ are independent

6、basis 基，其元素为基向量

V为向量空间，若r个向量 $a_1, a_2, \ldots a_r \in V$

且 $a_1, a_2, \ldots a_r$ 线性无关

且 V中任一向量都可由 $a_1, a_2, \ldots a_r$ 线性表示

则向量组 $a_1, a_2 \ldots a_r$ 称为 V 空间的一个基

r 为 V 空间的维数，并称 V 的 r 维向量空间

找基向量

主元列 自由列

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 0 \\ 1 & 3 & 1 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -5 & -1 & 0 \\ 0 & 1 & 2 & 1 & 0 \end{array}\right]$$

令 $x_3, x_4 = s$ 和 r $\quad x_1 = 5s + r$

$x_2 = -2s - r$

有 $\begin{bmatrix} 5s+r \\ -2s-r \\ s \\ r \end{bmatrix} = s\begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

a basis of V is $\begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

so $\dim V = 2$

(维度缩写，$\dim(A)$ = 维数)

rank(A) 秩数

Wronskian of function

朗斯基行列式可以用来确定一组函数在给定区间上的线性相关性。
对于n个n-1次连续可微函数f1、...、fn，它们的朗斯基行列式W(f1, ..., fn):

$$\begin{matrix} f_1 \\ f_2 \\ f_3 \end{matrix} \rightarrow \begin{bmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{bmatrix}$$

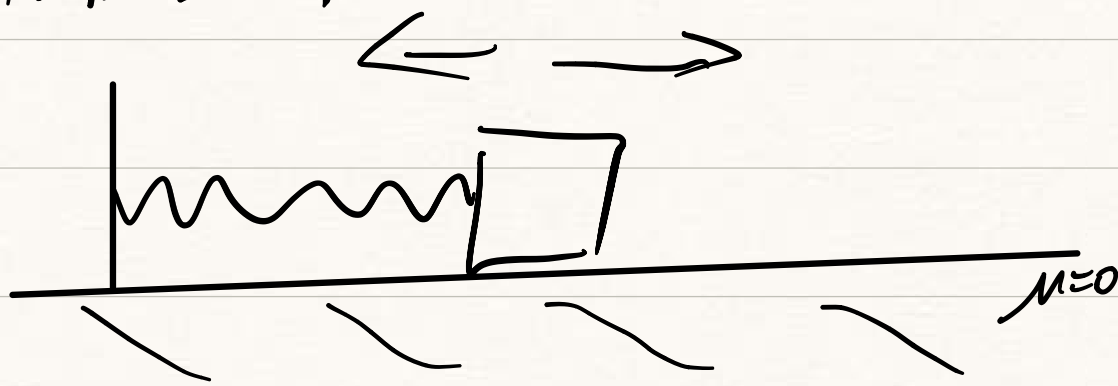
如果 f1、...、fn 在一个区间 [a, b] 上线性相关，则 W(f1, ..., fn) 在区间 [a, b] 上恒等于零。
也就是说，如果在某些点上 W(f1, ..., fn) 不等于零，则 f1、...、fn 线性无关

Exam1, Q4: $f_1(t)=1$, $f_2(t)=t$, $f_3(t)=e^t$

$$\begin{vmatrix} 1 & t & e^t \\ 0 & 1 & e^t \\ 0 & 0 & e^t \end{vmatrix} = e^t \neq 0$$

so it is linearly independent

@ 简谐运动 harmonic motion



设 $x = A\cos(\omega t+\varphi)$

已知 $x''+x=0$ 条 $x(0)=\dots$ $x'(0)=\dots$

求得原式 $x(t)\dots$

@二阶线性微分方程 $ay''+by'+cy=0$

$$\begin{cases} \text{两实根不等 } r_1 \neq r_2 \to y_h = C_1 e^{r_1 x} + C_2 e^{r_2 x} \\ \text{两实根相等 } r_1 = r_2 \to y_h = C_1 e^{r_1 x} + x \cdot C_2 e^{r_2 x} \\ \text{复根 } r_{1,2} = \alpha \pm i\beta \end{cases}$$

复杂的复数方程用公式解 $y_h = e^{\alpha x} \cdot (C_1 \cos\beta x + C_2 \sin\beta x)$

$r_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} \to \frac{-b}{2a} \pm i\frac{\sqrt{\Delta}}{2a}$

求y解.

Ex: $y''-6y'+9y=0$

相同根多个t

$r^2-6r+9=0$

$r_1=r_2=3$

令 $y(t) = C_1 e^{3t} + C_2 (t) e^{3t}$

$y(t)=C_1e \quad +C_2\, t\, e$

对于 $mx''+bx'+kx=0$ $\quad (m,k>0)$

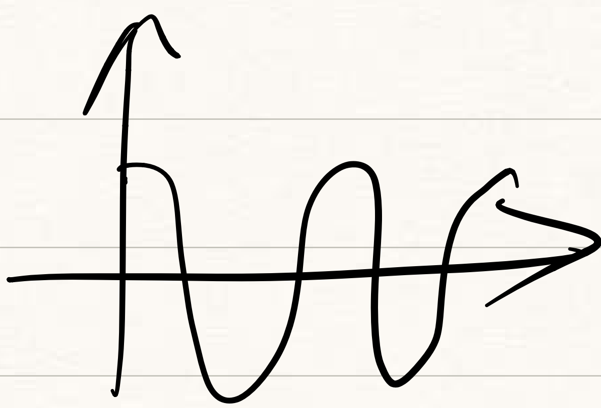
① 当 $b=0$ 时，为谐振运动

harmonic oscillator

$$x''=\frac{k}{m}x \rightarrow \omega_0^2=\frac{k}{m}$$

$$r^2=-\frac{k}{m}=-\omega_0^2$$

$$r=\pm\omega_0 i \rightarrow x(t)=C_1\cos\omega_0 t+C_2\sin\omega_0 t=A\cos(\omega_0 t+\varphi)$$



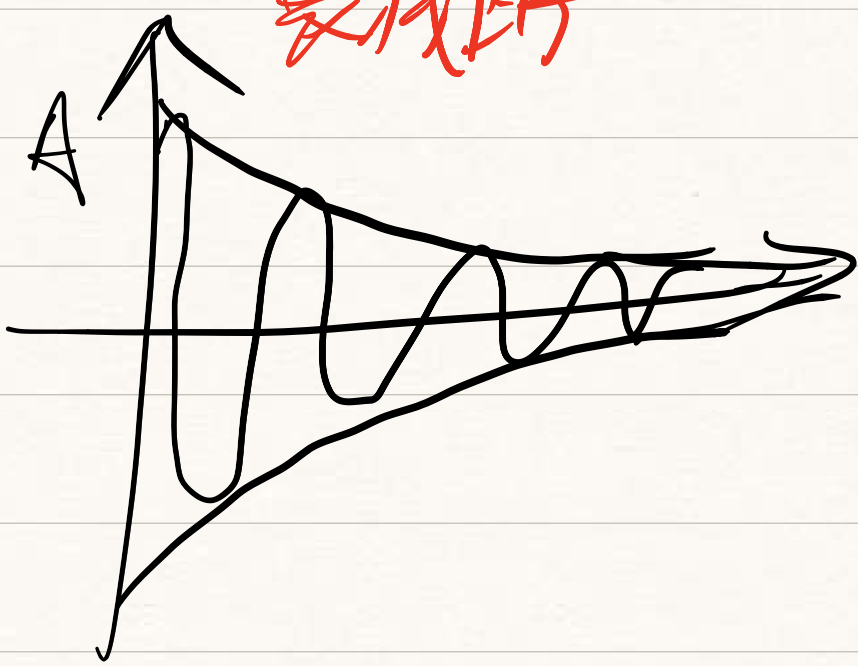
② 当 $b>0$ 时，$b^2<4km$ 时

$$r_{1,2}=\frac{-b\pm\sqrt{\Delta}}{2m}$$

$$=-\frac{b}{2m}\pm i\beta$$

$$x(t)=e^{-\frac{b}{2a}t}(C_1\cos\beta t+C_2\sin\beta t)$$

衰减因子



ex. $k=5\,N/m$, $0.5m$ rest position

Damping coefficient $b = 2\ \frac{Ns}{m}$
阻尼系数

停在0.5处放开

$\rightarrow \ddot{x} + 2\dot{x} + 5x = 0$, $x(0) = 0.5$, $x'(0) = 0$

$r^2 + 2r + 5 = 0$

$r = -1 \pm 2i$

$x(t) = e^{-t}(C_1\cos 2t + C_2\sin 2t)$

$\rightarrow x(t) = e^{-t}(\frac{1}{2}\cos 2t + \frac{1}{4}\sin 2t)$

通解找完还得求特解

Ex. $\boxed{2y'' + 3y' + y = t^2 - t + 2}$

通解 $\rightarrow y = C_1 e^{-\frac{1}{2}t} + C_2 e^{-t}$

设 $y_p = At^2 + Bt + C$，代入 y_p 原式

$$\begin{cases} y_p' = 2At + B \\ y_p'' = 2A \end{cases}$$

$4A + 6At + 3B + At^2 + Bt + C = t^2 - t + 2$

$At^2 + (6A + B)t + (4A + 3B + C) = t^2 - t + 2$

$\Rightarrow \{ A = 1$

$$\begin{cases} A = \\ B = -7 \\ C = 19 \end{cases}$$

$$y = y_h + y_p = C_1 e^{-\frac{1}{2}t} + C_2 e^{-t} + t^2 - 7t + 19$$

特解 y_p：

① $f(x)$ 为多项式，设次数一样的多项式 $P_n(x)$

设 $y_p = Q_n(x) = ax^2 + bx + c$ (2次)……

② $f(x)$ 为多项式 $\cdot e^{\alpha x}$ 形式，$P_n(x) \cdot e^{\alpha x}$

设 $y_p = e^{\alpha x} \cdot Q_n(x) \cdot x^k$

☆ $\begin{cases} k=0 & \alpha \neq r_1, \alpha \neq r_2 \\ k=1 & \alpha = r_1 \text{ 或 } \alpha = r_2 \\ k=2 & \alpha = r_1 = r_2 \end{cases}$

③ $f(x)$ 为 $e^{\alpha x}[P_m(x)\cos\beta x + P_n(x)\sin\beta x]$
三角形式

设 $y_p = e^{\alpha x}[Q_{1l}(x)\cos\beta x + Q_{2l}(x)\sin\beta x] \cdot x^k$

☆ l 为 m, n 的最高次数，共用 θ_2

$r = \alpha \pm \beta i$

$$* \begin{cases} k=0 & \alpha \pm \beta i \neq r_{1,2} \\ k=1 & \alpha \pm \beta i = r_{1,2} \end{cases}$$

另种方法

设 $v_1 y_1 + v_2 y_2$ 为特解

$$v_1 = -\int \frac{y_2 \cdot f(x)}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

$$v_2 = \int \frac{y_1 \ f(x)}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

Ex $y'' - 8y' + 15y = 8e^{5t}$

$y_1 = e^{3t},\ y_2 = e^{5t}$

$y_p = v_1 e^{3t} + v_2 e^{5t}$

@ 标准矩阵 standard matrix 'T'

设有一个 $\vec{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$T(x) = x_1 T(e_1) + x_2 T(e_2) = [T(e_1) \ T(e_2)]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$= Ax$$

也就是说对于每个线性变换 $T: R^n \to R^m$ 都有唯一个矩阵A，使得 $T(x) = Ax$，其中 $A = [T(e_1) \cdots T(e_n)]$

A被称为线性变换T的标准矩阵

$$A = \begin{bmatrix} 3 & 7 & 7 \\ -4 & -9 & 1 \end{bmatrix}, \ T: R^3 \to R^2$$

$T(\vec{x}) = A\vec{x}$, find the images of $\vec{u} = \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}$ & $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$\vec{x} = \begin{bmatrix} -1 & a \\ 5 & b \\ 1 & c \end{bmatrix}$$ 每列都是一个 x_i

$$A\vec{x} = \begin{bmatrix} 39 & 3a+7b+7c \\ -40 & -4a-9b+c \end{bmatrix}$$
$$= [T\vec{u} \ldots T\vec{v}]$$

$$\Rightarrow T(\vec{u}) = \begin{bmatrix} 39 \\ -40 \end{bmatrix}, \ T(\vec{v}) = \begin{bmatrix} 3a+7b+7c \\ -4a-9b+c \end{bmatrix}$$

☆ kernel 零空间，使 $T(v) = \vec{0}$ 的 v 集合，$\ker T = \{v \in V : T(v) = \vec{0}\}$ Ax=0解

image 列向量集合，参

$A=[v_1, v_2, \dots, v_m]$

Ker Im

nullity 零化度(零空间维度), rank 秩(像维度

$\operatorname{rank} A + \operatorname{nullity} A = n$

即 $\dim(\operatorname{Im}T)+\dim(\ker T)=\dim V$

injective 单射 surjective 满射 Bijective 双射

5.2

$T: \mathbb{R}^3 \to \mathbb{R}^3$

$T(x,y,z)=(x+2y+z,\ 2x+4y+z,\ x+y+z)$

Find matrix of T,

kernel of T, basis of it

images of T and basis of it

nullity of T, is T injective?

is T surjective?

解: $A = [\,T(e_1)\,|\,T(e_2)\,|\,T(e_3)\,] = \begin{bmatrix}1&2&1\\2&4&1\\1&1&1\end{bmatrix}$

$T(e_1) = T(1,0,0) = (1,2,1)$

$T(e_2) = T(0,1,0) = (2,4,1)$

$T(e_3) = T(0,0,1) = (1,1,1)$

$T(x,y,z) = \begin{bmatrix}1&2&1\\2&4&1\\1&1&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$\rightarrow x=0 \quad y=0 \quad z=0$

(Kernal即 令$A_x=0$集合) $\left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right]$

$\ker T = \{\vec{0}\}$, T is injective

nullity of $T = \dim(\ker T) = 0$

(if $\vec{u} \neq \vec{v}$ then $T(\vec{u}) \neq T(\vec{v})$)

$\text{im} = \left\{ c_1\begin{bmatrix}1\\2\\1\end{bmatrix} + c_2\begin{bmatrix}2\\4\\1\end{bmatrix} + c_3\begin{bmatrix}1\\1\\1\end{bmatrix},\ c_1, c_2, c_3 \text{ arbitrary} \right\}$

(image 列向量集合)

basis of im T is $\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\4\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}$

$\text{rank}T = \dim(\text{Im}\,T) = 3$

$\text{im}\,T = \mathbb{R}^3$, T surjective

$A\vec{x}=\vec{0}$

$$\left[\begin{array}{ccc|c}1&1&1&0\\1&2&1&0\\2&3&2&0\end{array}\right] \rightarrow \left[\begin{array}{ccc|c}1&0&1&0\\0&1&0&0\\0&0&0&0\end{array}\right]$$

解: $x+z=0$ $\quad$ z arbitrary (表任意)
$y=0$

$\ker T = \left\{ z\begin{bmatrix}-1\\0\\1\end{bmatrix}, z\in R \right\}$

basis of $\ker T$ is $\begin{bmatrix}-1\\0\\1\end{bmatrix}$

nullity of $\ker T = \dim \ker T = 1$

T is not injective

basis $\operatorname{Im} T = \operatorname{Span}\left\{\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$

$\operatorname{rank} T = \dim \operatorname{Im} T = 2$

$\operatorname{Im} T \neq R^3$, is not surjective

② 特征值和特征向量 λ

eigenvalue　　eigenvector

$$A\vec{x}=\lambda\vec{x}$$ ，λ为标量

得 $(A-\lambda I_n)\vec{x}=0$　$\vec{x}\neq 0$，

$$|A-\lambda I_n|=0$$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix}=0$$ ……可解

EX　$A=\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$　$\begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix}=(1-\lambda)^2-4=0$

$\lambda=-1$ 或 3

求特征向量有代回原阵，$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ 和 $\begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}$

$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=0$　　$\begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=0$

$x_1=-\frac{1}{2}x_2$　　$x_1=\frac{1}{2}x_2$

$\vec{v_1}=\begin{bmatrix} 1 \\ -2 \end{bmatrix}$　　$\vec{v_2}=\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

表示为 $E_{\lambda_1=-1}\left\{s\begin{bmatrix} 1 \\ -2 \end{bmatrix}, s\in\mathbb{R}\right\}$ $E_{\lambda_2=3}\left\{s\begin{bmatrix} 1 \\ 2 \end{bmatrix}, s\in\mathbb{R}\right\}$

$\dim E_\lambda \leq \lambda$ multiplicity 多样性

对应特征空间维数　　代数重数（重根数）

@ $\vec{x}' = A\vec{x} + \vec{F}$ 一阶微分方程矩阵版?

Ex: $\vec{x}' = A\vec{x}$, $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\lambda_1 = -1, \lambda_2 = 3$

则 $\begin{cases} x_1' = x_1 + 2x_2 \\ x_2' = 2x_1 + x_2 \end{cases}$ 有解 $u(t) = \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix}$

和 $v(t) = \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}$

验证 $Au = u'$, $Av = v'$ 确实

$|u|v| \neq 0$ 两线性无关, $x(t) = c_1 u(t) + c_2 v(t) = \begin{bmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

the fundamental matrix of $\vec{x}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \vec{x}$

is $W = [u(t)|v(t)] = \begin{bmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{bmatrix}$

先求 λ, 和对应特征向量 $\vec{v}$

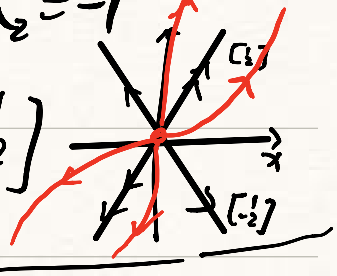
再 $\vec{x} = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2 \ldots$

$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Ex: $\vec{x}' = \begin{bmatrix} 1 & 1 \\ \ldots \end{bmatrix} \vec{x}$ $A = \begin{bmatrix} 1 & 1 \\ \ldots \end{bmatrix}$ 可得 $\lambda_1 = 3$, $\lambda_2 = \ldots$

Ex: $\vec{x}' = \begin{bmatrix} \cdots \\ 4 & 1 \end{bmatrix}\vec{x}$, $A = \begin{bmatrix} \cdots \\ 4 & 1 \end{bmatrix}$ 得 $\lambda_1 = 3, \lambda_2 = -1$

$\Rightarrow$ 解为 $c_1 e^{3t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$



- 两特征解相同时 $(\lambda_1 = \lambda_2 = \lambda)$ 即 $\dim E_\lambda = 1$ $\vec{x}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\vec{x}$ 同样下

得 $\begin{cases} (A - \lambda I_2)\vec{v} = 0 \\ (A - \lambda I_2)\vec{u} = \vec{v} \end{cases}$ 分别通过A求$\vec{v}$和$\vec{u}$

$$\vec{x} = c_1 e^{\lambda t}\vec{v} + c_2 e^{\lambda t}(t\vec{v} + \vec{u})$$

Ex: $\vec{x}' = \begin{bmatrix} 2 & -1 \\ 4 & 6 \end{bmatrix}\vec{x}$, $\vec{x}(0) = \begin{bmatrix} ? \\ \cdots \end{bmatrix}$

$$\begin{vmatrix} 2-\lambda & -1 \\ 4 & 6-\lambda \end{vmatrix} = (2-\lambda)(6-\lambda)+4 = \lambda^2-8\lambda+16$$

$$\lambda_1=\lambda_2=4$$

代入$\lambda=4$, $\begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=0$

$$\begin{cases} -2x-y=0 \\ 4x+y=0 \end{cases}$$

$$\vec{v}=\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$u+\begin{bmatrix} 0 \\ -1 \end{bmatrix}=\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ 一样的.

$\vec{u}$怎么得

特征阵

$$\begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}=\vec{v}=\begin{bmatrix} 1 \\ -2 \end{bmatrix} \Rightarrow \vec{u}=\begin{bmatrix} 0 \\ -1 \end{bmatrix} \text{或} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

则 $\vec{x}=c_1e^{4t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}+c_2e^{4t}\left(t\begin{bmatrix} 1 \\ -2 \end{bmatrix}+\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right)$

- complex eigenvalue 复数特征值 $x'=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\vec{x}$ 同样下

$$\boxed{\begin{array}{l}\text{欧拉公式 Euler's formula} \\ e^{\alpha+\beta i}=e^{\alpha}(\cos\beta+i\sin\beta)\end{array}}$$

$$\lambda=\alpha\pm\beta i$$

$$\vec{x}_c=e^{(\alpha+\beta i)t}\vec{v}=e^{\alpha}(\cos\beta t+i\sin\beta t)\vec{v}$$

$$\vec{x}=c_1\vec{x}_{Re}+c_2\vec{x}_{Im}$$ (i消失)

Ex: $\vec{x}' = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix} \vec{x}$

$$\begin{vmatrix} 6-\lambda & -1 \\ 5 & 4-\lambda \end{vmatrix} = (6-\lambda)(4-\lambda) + 5 = 0$$

$\lambda^2 - 10\lambda + 29 = 0$ $\left(\lambda = \frac{-b \pm \sqrt{\Delta}}{2a}\right)$

$(\lambda-5)^2 = -4$

$\lambda = 5 \pm 2i$

$\lambda = 5+2i$时 ($\lambda = 5-2i$结果一样)

$$\begin{bmatrix} 1-2i & -1 \\ 5 & -1-2i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

x设为1，$y = (1-2i)x$ 解$\vec{v} \begin{bmatrix} 1 \\ 1-2i \end{bmatrix}$

$$\vec{x}_c = e^{(5+2i)t} \cdot \begin{bmatrix} 1 \\ 1-2i \end{bmatrix}$$

$$= e^{5t}(\cos 2t + i \sin 2t) \begin{bmatrix} 1 \\ 1-2i \end{bmatrix}$$

$$= e^{5t} \begin{bmatrix} \cos 2t \\ \cos 2t + 2\sin 2t \end{bmatrix} + i e^{5t} \begin{bmatrix} \sin 2t \\ \sin 2t - 2\cos 2t \end{bmatrix}$$

$$\vec{x} = c_1 e^{5t} \begin{bmatrix} \cos 2t \\ \cos 2t + 2\sin 2t \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} \sin 2t \\ \sin 2t - 2\cos 2t \end{bmatrix}$$

ⓐ 稳定性和线性分类 stability and linear classification

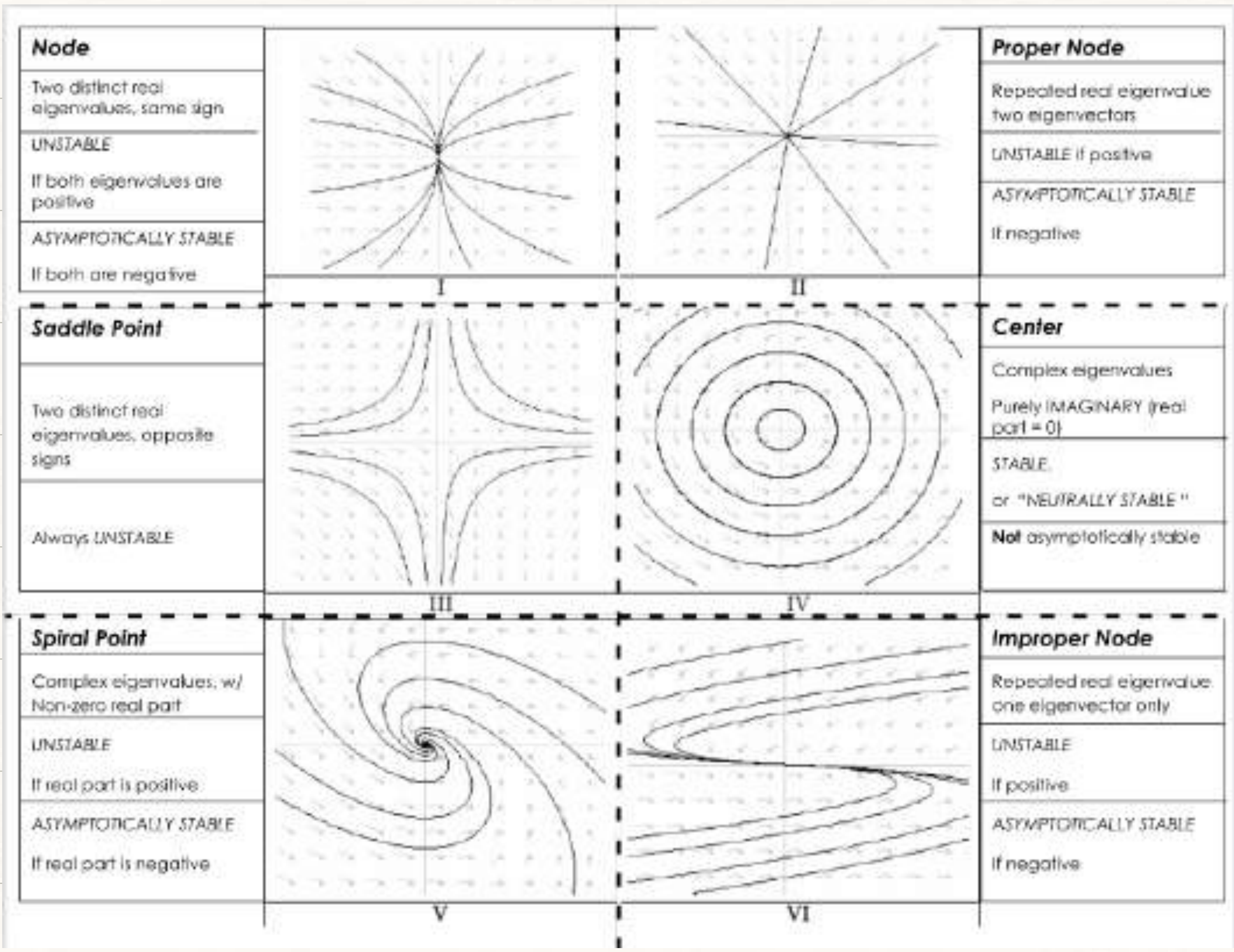
$\vec{x}' = F(x)$

$\Leftrightarrow \{ x' = f(x, y)$ $\vec{x}' =$

$\vec{x}=(x,y) \Longleftrightarrow \begin{cases} x' = f(x,y) \\ y'=g(x,y)\end{cases} \to \vec{x}'=A\vec{x}$ 求解

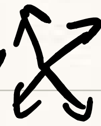
由 $\vec{x}(t)=c_1e^{\lambda t}\vec{v_1}+c_2e^{\lambda t}\vec{v_2}$ 可知 当 $0<\lambda<1$ 时 图象收缩至平衡点,
当 $\lambda>1$ 时 图象扩散

先令 $c_1=0$ 再 $c_2=0$ 找图象,再组合
由t↑和t↓画图



asymptotically stable 渐近于零 $(t\to\infty)$

neutrally stable 不会回到稳定点 但不无限远
不稳定 

unstable 无限远 

Node 集点扩散 

saddle 鞍型 

center 中心环绕  neutrally stable

spiral 旋儿 

@ Jacobian 雅可比

$$\begin{cases} x' = x - 3y + 2xy \ (= f) \\ y' = 4x - 6y - xy \ (= g) \end{cases}$$ 怎么求

$$J(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} 1+2y & -3+2x \\ 4-y & -6-x \end{bmatrix}$$

trace 对角线和

equilibium 平衡点

determinant 行列式

EX1

$$\begin{cases} x' = xy+9 \\ y' = 9x+y^3 \end{cases}$$

1 找平衡点:

$$\begin{cases} xy+9=0 \\ 9x+y^3=0 \end{cases}$$

$x = -\frac{1}{9}y^3 \quad -\frac{1}{9}y^4+9=0$

$y = \pm 3,\ x = \mp 3$

$(-3,3),\ (3,-3)$

$(-3,3)$点:

问题求 Jacobian matrix 的 trace, determinant in equilibrium

$$J_{(x,y)} = \begin{bmatrix} \frac{\partial\, xy+9}{\partial x} & \frac{\partial\, xy+9}{\partial y} \\ \frac{\partial\, 9x+y^3}{\partial x} & \frac{\partial\, 9x+y^3}{\partial y} \end{bmatrix} = \begin{bmatrix} y & x \\ 9 & 3y^2 \end{bmatrix}$$

$$J_{(-3,3)} = \begin{bmatrix} 3 & -3 \\ 9 & 27 \end{bmatrix}$$

$\text{trace} = 3+27=30$

对角线和

$\det = 3\cdot 27-(-3)9=108$

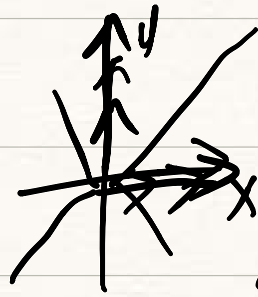
问题求 type of equilibrium is? and equilibrium is?

$$\begin{vmatrix} 3-\lambda & -3 \\ 9 & 27-\lambda \end{vmatrix} = (3-\lambda)(27-\lambda)+27=0$$

$$\lambda^2-30\lambda+108=0$$

$$\lambda = 15 \pm \sqrt{117}$$

$$\lambda_1, \lambda_2 > 0$$



saddle
unstable

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

(3, -3)点同理

$$J_{(x,y)} = \begin{bmatrix} y & x \\ 9 & 3y^2 \end{bmatrix} \quad J_{(3,-3)} = \begin{bmatrix} -3 & 3 \\ 9 & 27 \end{bmatrix}$$

trace=24

det $= -3\times 27 - 3\times 9 = -108$

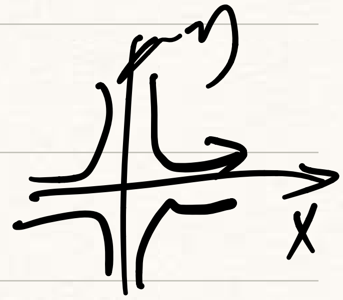
$$(-3-\lambda)(27-\lambda)-27=0$$

$$\lambda^2-24\lambda-108=0$$

$$(\lambda-12)^2=\sqrt{252}$$

$$\lambda = 12 \pm 15.8$$

$$\lambda_1 < 0 < \lambda_2$$



saddle

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

8.1 Laplace transform and inverse

拉普拉斯变换.

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{+\infty} f(t) \cdot e^{-st} dt$$

linear 线性, 加减乘除分开算

$\mathcal{L}\{1\} = \frac{1}{s}$ $\qquad \mathcal{L}^{-1}\{\frac{1}{s}\} = 1$

$\mathcal{L}\{t\} = \frac{1}{s^2}$ $\quad \mathcal{L}\{t^3\} = \frac{3!}{s^4}$ $\qquad \mathcal{L}^{-1}\{\frac{1}{s^2}\} = t$

$\mathcal{L}\{e^{-5t}\} = \frac{1}{s+5}$ $\qquad \mathcal{L}^{-1}\{\frac{1}{s+5}\} = e^{-5t}$

$$\mathcal{L}^{-1}\{\frac{4}{s^2} - \frac{3}{s+5}\} = 4\mathcal{L}^{-1}\{\frac{1}{s^2}\} - 3\mathcal{L}^{-1}\{\frac{1}{s+5}\}$$
$$= 4t - 3e^{-5t}$$

$\mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2}$ $\quad \mathcal{L}\{\cos bt\} = \frac{s}{s^2+b^2}$

原 $f(t)$	L变换
1 , $u(t)$	$\frac{1}{s}$ $(Res>0)$
t^n (t^4)	$\frac{n!}{s^{n+1}}$ $\left(\frac{4!}{s^5}\right)$ $(Res>0)$
$\sin wt$	$\frac{w}{s^2+w^2}$ $(Res>0)$
$\cos wt$	$\frac{s}{s^2+w^2}$ $(Res>0)$
e^{kt}	$\frac{1}{s-k}$ $(Res>0)$

$$\mathcal{L}^{-1}\left\{\frac{2s-14}{s^2-2s-3}\right\} = \mathcal{L}^{-1}\left\{\frac{4}{s+1}-\frac{2}{s-3}\right\}$$

$$= 4e^{-t}-2e^{3t}$$

$$\frac{2s-14}{s^2-2s-3} = \frac{2s-14}{(s+1)(s-3)} = \frac{A}{s+1}+\frac{B}{s-3}$$

$$\frac{2s-14}{s-3} = A + (s+1)\frac{B}{(s-3)}$$ 令$s=-1$求A

$$\frac{2s-14}{s+1} = (s-3)\frac{A}{s+1} + B$$ 令$s=3$求B

@拉普拉斯求微分方程

EX1 $y'-2y=e^{2t}$

$y(0)=3$

$Y(s)=\mathcal{L}\{y(t)\}$

思路：从高阶微分方程

$c_1y''+c_2y'+c_3y=\cdots$

$\downarrow \mathcal{L}$

变 $Y(s)=\cdots$

$\mathcal{L}^{-1}\downarrow \frac{A}{s-\lambda_1}+\frac{B}{s-\lambda_2}+\cdots$

$y(t)=\cdots$

$$sY(s)-y(0)-2Y(s)=\frac{1}{s-2}$$

$$(s-2)Y(s)=3+\frac{1}{s-2}$$

$$Y(s)=\frac{3}{s-2}+\frac{1}{(s-2)^2}\quad(s>2)$$

$$y(t)=3e^{2t}+te^{2t}$$

$$\mathcal{L}[y'(t)]=sY(s)-y(0)$$

$$\mathcal{L}[y''(t)]=s^2Y(s)-sy(0)-y'(0)$$

$$\mathcal{L}[y'''(t)]=s^3Y(s)-s^2y(0)-sy'(0)-y''(0)$$

Ex2

$$y''+5y'+4y=e^{-2t}$$

$$y(0)=0\ ,\ y'(0)=0$$

$$\mathcal{L}\rightarrow \underline{s^2Y(s)-sy(0)-y'(0)}+\underline{5sY(s)-5y(0)}$$

$$+4Y(s)=\frac{1}{s+2}$$

$$\Leftrightarrow s^2Y(s)+5sY(s)+4Y(s)=\frac{1}{s+2}$$

$$Y(s)(s+4)(s+1)=\frac{1}{s+2}$$

$$Y(s)=\frac{1}{(s+4)(s+1)(s+2)}$$

$$=\frac{\frac{1}{3}}{s+1}+\frac{-\frac{1}{2}}{s+2}+\frac{\frac{1}{6}}{s+4}$$

$$y(t)=\frac{1}{3}e^{-t}-\frac{1}{2}e^{-2t}+\frac{1}{6}e^{-4t}$$

Ex3:

$$y''' - y'' - y' + y = 6e^t$$

$$y(0) = y'(0) = y''(0)$$

$$\mathcal{L} \rightarrow \; s^3 Y(s) - s^2 y(0) - s y'(0) - y''(0)$$

$$- s^2 Y(s) + s y(0) + y'(0) + s Y(s) - y(0)$$

$$= (s^3 - s^2 + s) Y(s) = \mathcal{L}\{6e^t\}$$

$$= \frac{6}{s-1}$$

$$Y(s) = \frac{6}{(s-1)^3 (s+1)} = \frac{-\frac{3}{4}}{s+1} + \frac{\frac{3}{4}}{s-1} + \frac{-\frac{3}{2}}{(s-1)^2}$$

$$+ \frac{3}{(s-1)^3}$$

Ex4: $y''-10y'+41y=0,\ y(0)=0, y'(0)=4$

$$s^2Y(s)-sy(0)-y'(0)-10sY(s)+10y(0)$$
$$+41Y(s)=0$$

$$(s^2-10s+41)Y(s)-4=0$$

$$Y(s)=\frac{4}{s^2-10s+41}$$

$$Y(s)=\frac{4}{(s-5)^2+16} \qquad \frac{w}{s^2+w^2}\xrightarrow{\mathcal{L}^{-1}} \sin wt$$

$$y(t)=e^{5t}\cdot\sin 4t \qquad \frac{s}{s^2+w^2}\xrightarrow{\mathcal{L}^{-1}} \cos wt$$

遇多重根真分式 部分分式展开要求导

$$\frac{6}{(s-1)^3(s+1)}=\frac{A}{s+1}+\frac{B}{s-1}+\frac{C}{(s-1)^2}+\frac{D}{(s-1)^3}$$

单根 $s+1=0$时 $s=-1$　$\frac{6}{(s-1)^3}=A=-\frac{3}{4}$

多重根 $(s-1)^n=0$时 $s=1$　$\frac{6}{s+1}=B(s-1)^2+C(s-1)+D$

$$=D=3$$

$\downarrow$ 求导

$$-\frac{6}{(s+1)^2}=2B(s-1)+C$$

$$=C=-\frac{3}{2}$$

$\downarrow$ 求导

$$2\cdot\frac{6}{(s+1)^3}=2B=\frac{3}{2}\Rightarrow B=\frac{3}{4}$$

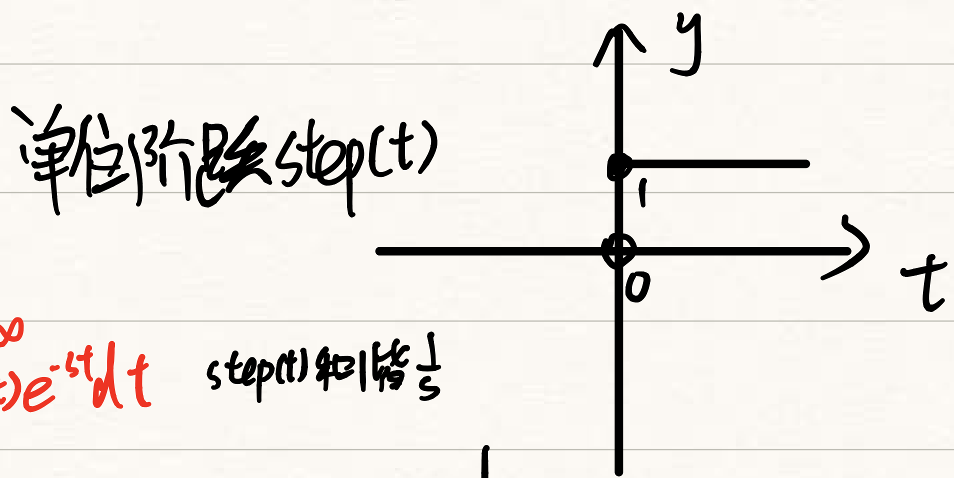
$$\Rightarrow Y(s)=\frac{6}{(s-1)^3(s+1)}=\frac{-\frac{3}{4}}{s+1}+\frac{\frac{3}{4}}{s-1}+\frac{-\frac{3}{2}}{(s-1)^2}+\frac{3}{(s-1)^3}$$

$\downarrow \mathcal{L}^{-1}$

$$y(t)=-\frac{3}{4}e^{-t}+\frac{3}{4}e^{t}-\frac{3}{2}te^{t}+\frac{3}{2}t^2e^{t}$$

② 阶跃函数 step(t) 的拉普拉斯变换

$$\begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$



$\mathcal{L}: \int_0^{+\infty} f(t)e^{-st}dt$ step(t)积分后 $\frac{1}{s}$

$\frac{1}{2\pi j}\int_0^{+\infty} F(s)e^{st}dt$

$u(t) \xrightarrow{\mathcal{L}} \frac{1}{s}$

$u(t-3) \xrightarrow{\mathcal{L}} \frac{1}{s}\cdot e^{-3s}$ 时移

$u(t+4) \xrightarrow{\mathcal{L}} \frac{1}{s}e^{4s}$

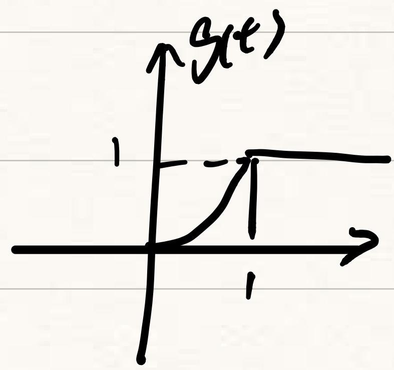
$u(t)\cdot e^{3t} \xrightarrow{\mathcal{L}} \frac{1}{s-3}$ 相移

$u(t)\cdot e^{-4t} \xrightarrow{\mathcal{L}} \frac{1}{s+4}$

Ex1: $f(t) = 2 - 6u(t-3) + 5u(t-4)$

$\mathcal{L}: \frac{2}{s} - \frac{6}{s}\cdot e^{-3s} + \frac{5}{s}\cdot e^{-4s}$

Ex2:



$$g(t) = \begin{cases} 0, & t < 0 \\ t^2, & 0 \leq t < 1 \\ 1, & t \geq 1 \end{cases}$$

$$g(t) = t^2(u(t)-u(t-1)) + u(t-1)$$
$$= t^2 u(t) + (1-t^2)u(t-1)$$

$$\mathcal{L}: \quad \frac{2!}{s^2} + \frac{1}{s}e^{-s} - \frac{2!}{s^2}\cdot e^{-s}$$

- <u>时相移动</u>: $f(t-a)\cdot u(t-a) \xrightarrow{\mathcal{L}} e^{-as}F(s)$

$$\underset{=f((t-a)+a)\cdot u(t-a)}{f(t)\cdot u(t-a)} \xrightarrow{\mathcal{L}} e^{-as}\mathcal{L}\{f(t+a)\}$$

Ex1: $(t-4)^3 u(t-4) \xrightarrow{\mathcal{L}} e^{-4s}F(s) = e^{-4s}\cdot\frac{3!}{s^4}$

$$= e^{-4s}\cdot\frac{6}{s^4}$$

Ex2: $x'' + x = u(t-4)$, $x(0)=0$, $x'(0)=1$

$$\mathcal{L}\{x''+x\} = \mathcal{L}\{u(t-4)\}$$

$$s^2X(s) - sX(0) - x'(0) + X(s) = e^{-4s}\cdot\frac{1}{s}$$

$$(s^2+1)X(s) - 1 = \frac{e^{-4s}}{s}$$

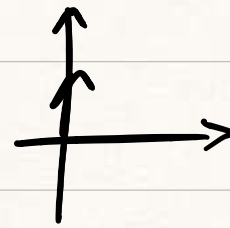
$$X(s) = \frac{1}{s^2+1} + \frac{1}{s(s^2+1)}\cdot e^{-4s}$$

$\frac{\quad}{s+1} \quad \frac{\quad}{(s+1)\cdot s}$

$$x(t)=\sin t+\mathcal{L}^{-1}\left\{\left(\frac{1}{s}-\frac{s}{s^2+1}\right)e^{-4s}\right\}$$
$$=\sin t+\left(1-\cos(t-4)\right)u(t-4)$$

② 单位冲激函数 unit impulse function

$$u'(t)=\delta(t)$$



面积为1

$$\mathcal{L}\{\delta(t)\}=1$$
$$\mathcal{L}\{\delta(t-a)\}=e^{-as}$$

Ex1: $y''+2y'+10y=\delta(t-2)$, $y(0)=1$, $y'(0)=0$

$\mathcal{L}$: $s^2Y(s)-sy(0)-y'(0)+2sY(s)-2y(0)+10Y(s)=e^{-2s}$

$$(s^2+2s+10)Y(s)=s+2+e^{-2s}$$

$$Y(s)=\frac{s+2}{s^2+2s+10}+\frac{e^{-2s}}{s^2+2s+10}$$
$$=\frac{1+s+1}{(s+1)^2+3^2}+\frac{1}{3}\cdot\frac{3}{(s+1)^2+3^2}\cdot e^{-2s}$$

$$= \frac{1}{3}\sin 3t \cdot e^{-t} + \cos 3t \cdot e^{-t} + \frac{1}{3}\sin 3(t-2) \cdot u(t-2) \cdot e^{-(t-2)}$$

$t<2$ 时 $y(t) = e^{-t}\left(\frac{1}{3}\sin 3t + \cos 3t\right)$

$t \geq 2$ 时 $y(t) = e^{-t}\left(\frac{1}{3}\sin 3t + \cos 3t\right) + \frac{1}{3}e^{-(t-2)}\sin 3(t-2)$

$t \to \infty$ 时两者相同

CH1 微分方程

CH2 一阶微分方程

CH3

CH4 二阶微分方程

$ay'' + by' + cy = f(x)$

y_p

CH5 矩阵 象ing 零空间ker . .

CH6 线性系统解

$\vec{x}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \vec{x}$

平衡点equilibrium (0,0)的类行和是否稳定stability

7.2

CH8 拉普拉斯变换

$y'' + by' + cy = f(t)$ $y(0) = \cdot\cdot$ $y'(0) = \cdot\cdot$

$\rightarrow Y(s) = \ldots$

$y(t) = \ldots$